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Constrained Guided Spiral Transition Curves

Zulfiqar Habib * and Manabu Sakai †
Kagoshima University, Japan ‡

Abstract

A method for drawing a guided G^2 continuous cubic spiral spline curve that falls within a closed boundary is presented. The boundary is composed of straight line segments and circular arcs. Spiral segments consist of transitions from straight line to straight line or circle. Guided curve can easily be controlled by shape control parameter. Our scheme has better smoothness and more degree of freedom than any previous method. Also our scheme is completely local and hence more suitable and comfortable for practical use.
Key words: cubic, guided, G^2 spiral, constrained spline

1 Introduction

A method for drawing a guided G^2 continuous cubic spiral spline curve that falls within a closed boundary is presented. The boundary is composed of straight line segments and circular arcs. We discuss cubic G^2 spiral transition from straight line to circle. Then we extend it for transition between two non-parallel straight lines and finally make it suitable for constrained guided curve. Our constrained curve can easily be controlled by shape control parameter. Any change in this shape control parameter does not effect the continuity and neighborhood parts of the curve. There are several problems whose solution requires these types of methods. For example

- A user may wish to design a curve that fits inside a given region as, for example, when one is designing a shape to be cut from a flat sheet of material.
- A user may wish to design a smooth path that avoids obstacles as, for example, when one is designing a robot or auto drive car path.
- For applications such as the design of highways or railways it is desirable that curve be fair. In the discussion about geometric design standards in AASHO (American Association of State Highway Officials), Hickerson [6] (p. 17) states that "Sudden changes between curves of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of *gradually increasing or decreasing radii* without at the same time introducing an appearance of forced alignment". The importance of this design feature is highlighted in [3] that links vehicle accidents to inconsistency in highway geometric design.

Parametric cubic curves are popular in CAD applications because they are the lowest degree polynomial curves that allow inflection points (where curvature is zero), so they are suitable for the composition of G^2 splines. To be visually pleasing it is desirable that the spline be fair. The Bézier form of a parametric cubic curve is usually used in CAD/CAM and CAGD (Computer Aided Geometric Design) applications because of its geometric and numerical properties. Many authors have advocated their use in different applications like data fitting and font designing. The importance of using fair curves in the design process is well documented in the literature [2]. Cubic curves, although smoother, are not always helpful since they might have unwanted inflection points and singularities (See [10]). Spirals have several advantages of containing neither inflection points, singularities nor curvature extrema (See [5]). Such curves are useful for transition between two circles or straight lines.

*E-mail: drzhabib@hotmail.com; Web: http://zulfiqar.8m.com
†E-mail: msakai@sci.kagoshima-u.ac.jp, Department of Mathematics and Computer Science
‡Graduate School of Science and Engineering, Korimoto 1-21-35, Kagoshima University, Kagoshima 890-0065, Japan.

Many authors have discussed the problem of drawing constrained curves. In [4], a G^2 continuous, shape-preserving curve made of rational cubics that interpolates to given points and that lies on one side of a line, or several lines, is described. In [8], fair Bézier curves with fixed cross-sectional area are produced. In [1], a C^1 continuous non-parametric interpolation rational (cubic numerator and linear denominator) that lies above, below, or between polylines is discussed. In [9], a G^2 continuous curve made of non-parametric rational cubics that lies on one side of a line, or one side of a quadratic curve is found.

In our paper, the boundary consists of straight line segments and circular arcs, which is different than boundaries considered by above mentioned papers. Recently, Meek [7] presented a method for drawing a guided G^1 continuous planar spline curve that falls within a closed boundary composed of straight line segments and circles. G^1 continuity is not suitable for many practical applications where high degree of smoothness is required, for example, high way designing. Our guided curve has better smoothness than Meek [7] scheme which has G^1 continuity. The objectives and shape features of our scheme in this paper are

- To obtain G^2 cubic spiral transition from straight line to circle and make it more flexible than Walton scheme in [11].
- To obtain cubic spiral transition between two non-parallel straight lines.
- To obtain guided G^2 continuous cubic spiral spline that falls within a closed boundary composed of straight line segments and circles.
- To discuss and prove shape features of cubic spiral.
- To achieve more degrees of freedom and flexible constraints for easy use in practical applications.
- Any change in shape parameter does not effect continuity and neighborhood parts of our guided spline. So, our scheme is completely local.

In this paper, $\times$ stands for the two-dimensional cross product $(x_0, y_0) \times (x_1, y_1) = x_0y_1 - x_1y_0$ and $\|\bullet\|$ means the Euclidean norm. Let L be a straight line through origin O and a circle Ω of radius r centered at C. Consider the planar curve $z(t) = (x(t), y(t)), 0 \leq t \leq 1$ and for later use, consider

$$u(t) = u_0(1-t)^2 + 2u_1t(1-t) + u_2t^2, \quad v(t) = v_0(1-t)^2 + 2v_1t(1-t) + v_2t^2 \tag{1.1}$$

A spiral is a curve whose curvature does not change sign and whose curvature is monotone. G^2 (Geometric continuity of second order) means continuity in position, in unit tangent, and in signed curvature. A curve is said to match G^2 Hermite data if it passes from one given point to another given point, if its unit tangent matches given unit tangents at the two given points, and its signed curvature matches given signed curvatures at the two given points.

This paper does not include discussion on PH quintic spiral due to page limitation, however it will appear soon in our next paper. The organization of our paper is as follows. We start from G^2 cubic Bézier function, then description of method for spiral transition from straight line to circle, its extension for transition between nonparallel straight lines and finally application to constrained guided spline. Numerical examples, analysis, comparison and conclusions are given in last section.

2 Spiral Transition From Straight Line to Circle

We consider a cubic transition $z(t)$ (see Figure 1) of the form $z'(t) = (u(t), v(t))$. Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) \left(= \frac{z'(t) \times z''(t)}{\|z'(t)\|^3} \right) = \frac{u(t)v'(t) - u'(t)v(t)}{\{u(t)^2 + v^2(t)\}^{3/2}} \tag{2.1}$$

For later use, consider

$$\begin{aligned} \{u^2(t)+v^2(t)\}^{5/2}\kappa'(t) &= \{u(t)v''(t) - u''(t)v(t)\}\{u^2(t)+v^2(t)\} \\ &-3\{u(t)v'(t) - u'(t)v(t)\}\{u(t)u'(t)+v(t)v'(t)\} \ (= w(t)) \end{aligned} \tag{2.2}$$

Here, we require for $0 < \theta < \pi/2$

$$z(0) = (0,0), \quad z'(0) \parallel (1,0), \quad \kappa(0) = 0, \quad z'(1) \parallel (\cos\theta, \sin\theta), \quad \kappa(1) = 1/r \tag{2.3}$$

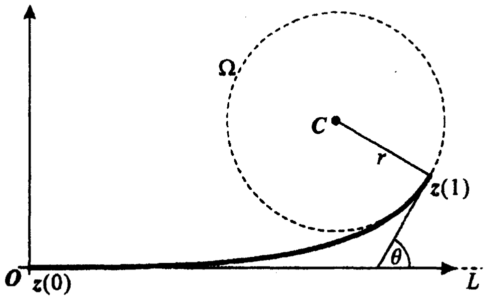


Figure 1: Cubic spiral transition from straight line to circle.

Then, the above conditions require

Lemma 2.1. *With a positive parameter d*

$$u_1 = \frac{d^2}{2r\sin\theta}, \quad u_2 = d\cos\theta, \quad v_0 = 0, \quad v_1 = 0, \quad v_2 = d\sin\theta \tag{2.4}$$

where $z'(0) = (u_0, 0)$ *and* $z'(1) = d(\cos\theta, \sin\theta)$.

We introduce a pair of parameters (m, q) for (u_0, d) as $u_0 = mu_1, d = qr$, Then

$$u_0 = \frac{mrq^2}{2\sin\theta}, \quad u_1 = \frac{rq^2}{2\sin\theta}, \quad u_2 = qr\cos\theta, \quad v_0 = v_1 = 0, \quad v_2 = qr\sin\theta \tag{2.5}$$

from which

$$x(t) = \frac{qrt}{6\sin\theta}\left[q\left\{(3-2t)t + m(3-3t+t^2)\right\} + t^2\sin 2\theta\right], \quad y(t) = \frac{qrt^3\sin\theta}{3} \tag{2.6}$$

Walton ([11]) considered a cubic curve $z(t)(= (x(t), y(t))), 0 \le t \le 1$ of the form

$$z(t) = b_0(1-t)^3 + 3b_1t(1-t)^2 + 3b_2(1-t)t^2 + b_3t^3 \tag{2.7}$$

where Bézier points $b_i, 0 \le i \le 3$ are defined as follows

$$b_1 - b_0 = b_2 - b_1 = \frac{25r\tan\theta}{54\cos\theta}(1,0), \quad b_3 - b_2 = \frac{5r\tan\theta}{9}(\cos\theta, \sin\theta) \tag{2.8}$$

Simple calculation gives

$$x'(t)(= u(t)) = u_0(1-t)^2 + 2u_1t(1-t) + u_2t^2, \quad y'(t)(= v(t)) = v_0(1-t)^2 + 2v_1t(1-t) + v_2t^2 \tag{2.9}$$

where

$$u_0 = u_1 = \frac{25r\tan\theta}{18\cos\theta}, \quad u_2 = \frac{5r\sin\theta}{3}, \quad v_0 = v_1 = 0, \quad v_2 = \frac{5r\tan\theta\sin\theta}{3} \tag{2.10}$$

Hence, note that their method is our special case with $m = 1$ and $q = \frac{5}{3}\tan\theta$.

With help of a symbolic manipulator, we obtain

$$\kappa'\left(\frac{1}{1+s}\right) = \frac{8(1+s)^5\left(\sum_{i=0}^{5} a_is^i\right)\sin^3\theta}{r\left\{q^2s^2(2+ms)^2 + 2qs(2+ms)\sin 2\theta + 4\sin^2\theta\right\}^{5/2}} \tag{2.11}$$

where

$$\begin{aligned}
&a_0 = 4\left\{3q\cos\theta - (4+m)\sin\theta\right\}\sin\theta, \quad a_1 = 2\left\{6q^2 - q(5-4m)\sin 2\theta - 10m\sin^2\theta\right\} \\
&a_2 = 2q\left\{(-2+13m)q - 2m(4-m)\sin 2\theta\right\}, \quad a_3 = 2mq\left\{(-3+10m)q - 2m\sin 2\theta\right\} \\
&a_4 = 5m^3q^2, \quad a_5 = m^3q^2
\end{aligned}$$

Hence, we have a sufficient spiral condition for a transition curve $z(t), 0 \le t \le 1$, i.e., $a_i \ge 0, 0 \le i \le 5$

Lemma 2.2. *The cubic segment $z(t), 0 \le t \le 1$ of the form (1.1) is a spiral satisfying (2.4) if $m > 3/10$ and*

$$q \ge q(m,\theta)\left(= Max\left[\frac{(4+m)\tan\theta}{3}, \frac{2m(4-m)\sin 2\theta}{13m-2}, \frac{2m\sin 2\theta}{10m-3},\right.\right.$$
$$\left.\left.\frac{1}{6}\left\{(5-4m)\cos\theta+\sqrt{60m+(5-4m)^2\cos^2\theta}\right\}\sin\theta\right]\right) \tag{2.12}$$

Theorem 2.1. *The cubic segment $z(t), 0 \le t \le 1$ of the form (1.1) is a spiral satisfying (1.1) and $\kappa'(1) = 0$ for $\theta \in (0, \pi/2]$ if $m \ge 2(-1+\sqrt{6})/5 (= c_0)(\approx 0.5797)$*

Proof. Letting $z = \tan\theta (> 0)$, we only have to note that the terms in brackets of (2.11) reduce

$$\frac{(4+m)z}{3}(= A_1), \frac{4m(4-m)z}{(13m-2)(1+z^2)}(= A_2), \frac{4mz}{(10m-3)(1+z^2)}(= A_3),$$
$$\frac{z\left\{5-4m+\sqrt{25+16m^2+20m(1+3z^2)}\right\}}{6(1+z^2)}(= A_4)$$

Here, we have to check that the first quantity is not less than the remaining three ones where

$$A_1 \ge A_2 \quad (m \ge \frac{1}{25}(-1+\sqrt{201})(\approx 0.5270))$$
$$A_1 \ge A_3 \quad (m \ge \frac{1}{20}(-25+\sqrt{1105})(\approx 0.4120)), \quad A_1 \ge A_4 \quad (m \ge c_0)$$

□

3 Family of Spiral Transitions Between Two Straight Lines

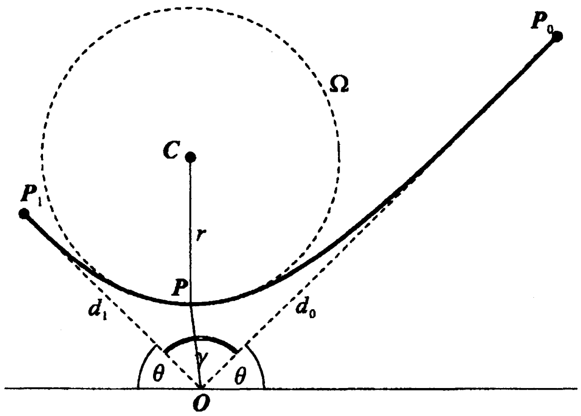


Figure 2: Cubic spiral transition between two straight lines.

Here, we have extended the idea of straight line to circle transition and derived a method for Bézier spiral transition between two nonparallel straight lines (see Figure 2). We note the following result that is of use for joining two lines. For $0 < \theta < \pi/2$, we consider a cubic curve satisfying

$$z(0) = (0,0), \quad z'(0) \parallel (1,0), \quad \kappa(0) = 1/r, \quad z'(1) \parallel (\cos\theta, \sin\theta), \quad \kappa(1) = 0 \tag{3.1}$$

Since Bézier curves are affine invariant, a cubic Bézier spiral can be used in a coordinate free manner. Therefore transformation, i.e., rotation, translation, reflection with respect to y-axis and change of variable t with $1-t$ to (2.6) gives $z(t)(= (x(t), y(t)))$ by

$$x(t) = \frac{qrt}{6\sin\theta}\left[qt\{3-(2-m)t\}\cos\theta + 2(3-3t+t^2)\sin\theta\right], \quad y(t) = \frac{q^2rt^2}{6}\{3-(2-m)t\} \tag{3.2}$$

Now, $z(t,m,\theta)(=(x(t,m,\theta),y(t,m,\theta))), 0 \le t \le 1$ denotes the cubic spline satisfying (3.1). Assume that the angle between two lines is γ $(<\pi)$. Then, $(x(t,m,\theta_0),y(t,m,\theta_0))$ of the form (3.2) with $q = q_0(\ge q(m,\theta_0))$ and $(-x(t,n,\theta_1),y(t,n,\theta_1))$ of the form (3.2) with $q_1(\ge q(n,\theta_1))$ with $\theta_0+\theta_1 = \pi-\gamma$ is a pair of spiral transition curves. Figures 3(a, b) shows the graphs of the family of G^2 cubic spiral transition curves between two straight lines for $(\gamma,\theta_0,\theta_1) = (5\pi/12, \pi/4, \pi/3)$, (a) $(m_0, m_1) = (3, 0.8)$ (shaded) and (1, 1) (bold), (b) $r = 0.1$ (shaded) and 0.2 (bold). Here start and end points of straight lines are not fixed and user has no control over them. This situation is not suitable for some practical applications.

3.1 Scheme For Fixed End Points

When $\theta_0 = \theta_1 (= \theta = (\pi-\gamma)/2)$ are fixed and $m, n \ge c_0$ (then, note $q_0 = (4+m)/3\tan\theta$ and $q_1 = (4+n)/3\tan\theta$), the distances between the intersection O of the two lines and the end points $P_i, i = 0,1$ of the transition curves are given by $d(m,n,r,\theta)$ and $d(n,m,r,\theta)$ where

$$d(m,n,r,\theta) = (r/c)\left\{4+(3+m)^3+3(3+n)\right\}, \quad c = 54\cos^2\theta/\sin\theta \tag{3.3}$$

Here we derive a condition on r for which the following system of equations has the solutions $m, n(\ge c_0)$ for given nonnegative d_0, d_1

$$d(m,n,r,\theta) = d_0, \quad d(n,m,r,\theta) = d_1 \tag{3.4}$$

Let $(\alpha,\beta) = (3+m, 3+n)$ reduce the above system to

$$\alpha^3 + 3\beta = \lambda(= cd_0/r - 4), \quad \beta^3 + 3\alpha = \mu(= cd_1/r - 4) \tag{3.5}$$

where require $m, n \ge c_0$ to note $\alpha, \beta \ge c_1(= c_0+3)$ and $\lambda, \mu \ge c_2(= c_1^3 + 3c_1)$. Delete α from (1.23) to get a quartic equation $f(\beta) = 0$

$$f(\beta) = \beta^9 - 3\mu\beta^6 + 3\mu^2\beta^3 - 81\beta + 27\lambda - \mu^3 \tag{3.6}$$

Restrictions: $\alpha, \beta \ge c_1$ require that at least one root β of $f(\beta) = 0$ must satisfy

$$c_1 \le \beta \le (\mu - 3c_1)^{1/3} \tag{3.7}$$

Intermediate value of theorem gives a sufficient condition: $f(c_1) \le 0$ and $f((\mu-3c_1)^{1/3}) \ge 0$ where

$$\begin{aligned} f(c_1) &= -\mu^3 + 3c_1^3\mu^2 - 3c_1^6\mu + 27\lambda + c_1^9 - 81c_1 = -\left\{\mu - c_1^3 - 3(\lambda-3c_1)^{1/3}\right\} \times \\ &\left[(\mu-c_2)^2 + 3\left\{2c_1 + (\lambda-3c_1)^{1/3}\right\}(\mu-c_2) + 9\left\{c_1^2 + c_1(\lambda-3c_1)^{1/3} + (\lambda-3c_1)^{2/3}\right\}\right] \\ f((\mu-3c_1)^{1/3}) &= 27\left\{\lambda - c_1^3 - 3(\mu-3c_1)^{1/3}\right\} \end{aligned} \tag{3.8}$$

Since the quantity in brackets is positive for $\mu \ge c_2$, the sufficient one reduces to

$$\lambda - c_1^3 \ge 3(\mu-3c_1)^{1/3}, \quad \mu - c_1^3 \ge 3(\lambda-3c_1)^{1/3} \tag{3.9}$$

Note $\lambda, \mu \ge c_2$ to obtain

Lemma 3.1. *Given d_0, d_1, assume that r satisfies $\lambda - c_1^3 \ge 3(\mu-3c_1)^{1/3} \ge 0, \mu - c_1^3 \ge 3(\lambda-3c_1)^{1/3} \ge 0$. Then the system of (3.5) has the required solutions α, β $(\ge c_1)$.*

Note $\lambda, \mu = O(1/r), r \to 0$ to obtain that a small value of r makes the inequalities (3.9) be valid for any d_0, d_1. Next, for an upper bound for r, we require

Lemma 3.2. *If $d_0 \ge d_1$, then $\mu - c_1^3 = 3(\lambda-3c_1)^{1/3}$ and $\lambda, \mu \ge c_2$ by (3.5) has a unique positive solution r^*.*

Proof. Let $t = 1/r$ to reduce $\mu - c_1^3 = 3(\lambda - 3c_1)^{1/3}$ to

$$f(t)\ \left(= (cd_1 t - 4 - c_1^3)^3 - 27(cd_0 t - 4 - 3c_1)\right) = 0 \tag{3.10}$$

where $\lambda, \mu \geq c_2$ are equivalent to $t \geq (c_2 + 4)/(cd_1)$. First, note with $d_0 = k^2 d_1$ $(k \geq 1)$

$$f(+\infty) = +\infty, \quad f\left(\frac{c_2 + 4}{cd_1}\right) = -27(c_1 + 1)(c_1^2 - c_1 + 4)(k^2 - 1)(\leq 0)$$

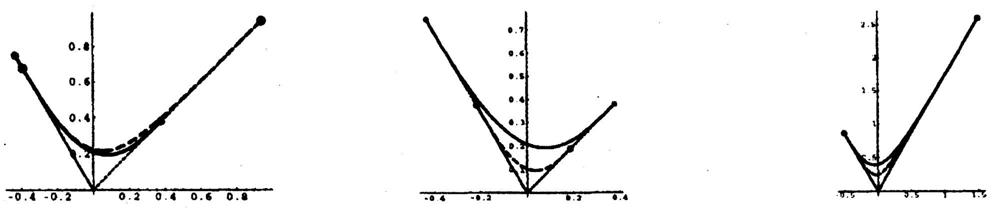
In addition, $f(t)$ has its relative maximum at $t = (c_1^3 + 4 - 3k)/(cd_1)\,(< (c_2 + 4)/(cd_1))$. Therefore, $f(t) = 0$ has just one root $t = t^*(= 1/r^*)\ (\geq (c_2 + 4)/(cd_1))$. □

As r increases from zero, "equality" in the second inequality of (3.9) is first valid. Hence, we obtain

Theorem 3.1. *Assume that $d_0 \geq d_1$. Then the system of equation (3.4) in m, n $(\geq c_0)$ is solvable for $0 < r \leq r^*$ where r^* is the positive root no greater than $cd_1/(c_2 + 4)$*

$$\left\{(4 + c_1^3)r - cd_1\right\}^3 - 27r^2\left\{(4 + 3c_1)r - cd_0\right\} = 0, \quad c_1 = c_0^3 + 3c_0 \tag{3.11}$$

Then, for the angle γ $(< \pi)$ between the two straight lines with $\theta = (\pi - \gamma)/2$, $(x(t, m, \theta),\ y(t, m, \theta))$, $q = \{(4 + m)/3\}\tan\theta$ of the form (3.2) and $(-x(t, n, \theta),\ y(t, n, \theta))$, $q = \{(4 + n)/3\}\tan\theta$ of the form (3.2) is a pair of spiral transition curves between the two lines.



(a) Given $r = 0.2$ (b) Given $(m, n) = (1, 1)$ (c) Given $(d_0, d_1) = (1, 3)$

Figure 3: Graphs of $z(t)$ with non-fixed (a, b) and fixed (c) start and end points.

This result enables the pair of the spirals to pass through the given points of contact on the nonparallel two straight lines. For example, in Figure 3(c), straight lines are given with $\gamma = \theta_0 = \theta_1 = \pi/3$, $r = 0.1$ (shaded) and 0.2 (bold). By (3.4), for $r = 0.1$, $(m_0, m_1) \approx (4.65, 2.05)$ and for $r = 0.2$, $(m_0, m_1) \approx (3.02, 0.82)$. To keep the transition curve within a closed boundary, value of shape control parameter r can be derived from (3.2) when control points and boundary information are given. A constrained guided curve is shown in Figure 4.

4 Examples, Analysis and Conclusion

In Figure 4, a constrained G^2 continuous cubic spiral spline uses straight line to straight line transition. The boundary is composed of straight line segments and circular arcs. These curves are guided by shape control parameter and all segments have completely local control, i.e., any change in one segment does not effect continuity and shape of neighboring segments. The data for boundary and control points has been taken from Figure 8 in [7] which has G^1 continuity and scheme is not completely local.

Cubic spiral segments are useful in the design of objects when it is desirable that fair curves be used in the design process. A method for straight line to circle transition has been discussed and extended to transition curve between two non parallel straight lines. We proved that Walton scheme for cubic [11] is a special case of our most flexible scheme. We offered reasonable degree of freedom and extended our schemes to G^2 guided spiral spline constrained by a closed boundary. Our scheme has G^2 continuity which is better than G^1 continuity in [7]. G^1 continuity is not suitable for many practical applications where high degree of smoothness is required.

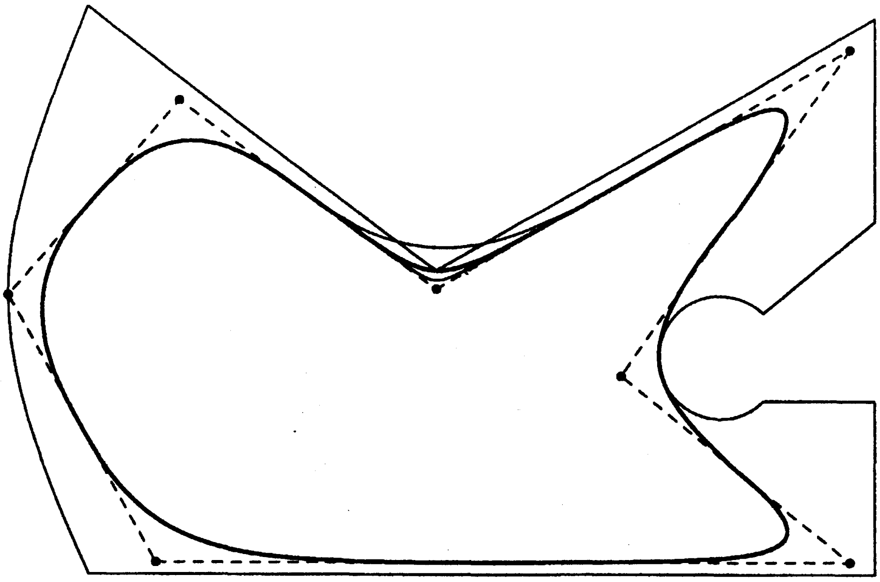


Figure 4: A G^2 cubic guided spiral spline constrained by a closed boundary.

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